Partial Solutions to Colin Doyle Mathematics Contest 1980

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I started with the questions most likely to cause difficulties. I'll type up more when I have some spare time.

Question 2

(b) Since m^4 and $4n^4$ are both perfect squares, we can try "completing the square" and see if that gets us anywhere useful.

$$\begin{aligned} m^4 + 4n^4 &= (m^2)^2 + (2n^2)^2 \\ &= (m^2)^2 + 2(m^2)(2n^2) + (2n^2)^2 - 2(m^2)(2n^2) \\ &= (m^2 + 2n^2)^2 - (2mn)^2 \\ &= (m^2 - 2mn + 2n^2)(m^2 + 2mn + 2n^2) \end{aligned}$$

The set of integers in closed under addition, subtraction and multiplication. Since m and n are integers, this implies $m^2 - 2mn + 2n^2$ and $m^2 + 2mn + 2n^2$ are also integers. That is, $m^4 + 4n^4$ is the product of two integers, but it could still be prime if one of those integers happens to be 1, so we should check the minimum possible values of those two integers.

We know $m > 1$ and $n > 1$. Hence

$$m^2 - 2mn + 2n^2 = (m - n)^2 + n^2 > 0 + 1^2 = 1$$

and

$$m^2 + 2mn + 2n^2 > 1^2 + 2 \times 1 \times 1 + 2 \times 1^2 = 5$$

That is, given the restrictions on m and n, $m^4 + 4n^4$ is the product of two integers both of which exceed 1, and hence it is never prime.

Question 3

(a) Let d denote the common difference.

Let u_n denote the n^{th} term of the arithmetic progression.

(Depending which textbook you learned from, you may be used to using T_n for the n^{th} term, but the question has defined T_n to mean something else, so we need to use some other symbol for the n^{th} term.)

Method 1

If $\dfrac{T_n}{S_n}$ is independent of n, then we can simply equate any two cases of that expression that use particular values of n. To keeps things easy, choose the simplest values possible.

$$\begin{aligned}
\frac{T_1}{S_1} &= \frac{T_2}{S_2} \\
\frac{u_2}{u_1} &= \frac{u_3 + u_4}{u_1 + u_2} \\
\frac{2+d}{2} &= \frac{(2+2d)+(2+3d)}{2+(2+d)} \\
&= \frac{4+5d}{4+d} \\
(2+d)(4+d) &= 2(4+5d) \\
d^2 + 6d + 8 &= 10d + 8 \\
d^2 - 4d &= 0 \\
d(d-4) &= 0 \\
d &= 0, 4
\end{aligned}$$

Since the question states that the common difference is not zero, it must be 4.

Note that the algebra we have performed does not prove that when $d = 4$ the fraction $\frac{T_n}{S_n}$ will be independent of n. But the question states that there is a non-zero common difference which causes that fraction to be independent of n. Given that fact, our algebra shows that the only possible non-zero value for that common difference is 4. Thus we have produced a valid answer.

But if you have spare time, it would be worth checking that $d = 4$ does produce the desired property, just to be sure we haven't made an algebra error somewhere.

The check

$u_n = 2 + (n-1) \times 4 = 4n - 2$

We can use either of the two common formulae for summing the arithmetic progression. This solution will use the formulae that employs the last term.

$$\begin{aligned}
\frac{T_n}{S_n} &= \frac{u_{n+1} + u_{n+2} + \cdots + u_{2n}}{u_1 + u_2 + \cdots + u_n} \\
&= \frac{\frac{n}{2}(u_{n+1} + u_{2n})}{\frac{n}{2}(u_1 + u_n)} \\
&= \frac{u_{n+1} + u_{2n}}{u_1 + u_n} \\
&= \frac{4(n+1) - 2 + 4(2n) - 2}{2 + 4n - 2} \\
&= \frac{12n}{4n} \\
&= 3
\end{aligned}$$

The result is independent of n, as required.

Method 2

Method 1 was an insightful solution that realised we could employ particular values of n to deduce the common difference. Method 2 won't use that insight. In method 2 the algebra will resemble the check in method 1, but it's harder since we have to track the common difference as a pronumeral rather than just checking for a particular value of that pronumeral.

$u_n = 2 + (n-1)d$

Repeat the first 3 lines of the evaluation in the check above and then proceed as follows.

$$\begin{aligned}
\frac{T_n}{S_n} &= \frac{u_{n+1} + u_{2n}}{u_1 + u_n} \\
&= \frac{u_1 + nd + u_n + nd}{u_1 + u_n} \\
&= 1 + \frac{2nd}{u_1 + u_n} \\
&= 1 + \frac{2nd}{2 + 2 + (n-1)d} \\
&= 1 + \frac{2nd}{4 + nd - d} \\
&= 1 + \frac{(8 + 2nd - 2d) + (2d - 8)}{4 + nd - d} \\
&= 1 + 2 + \frac{2d - 8}{4 + nd - d} \\
&= 3 + \frac{2(d-4)}{4 + nd - d}
\end{aligned}$$

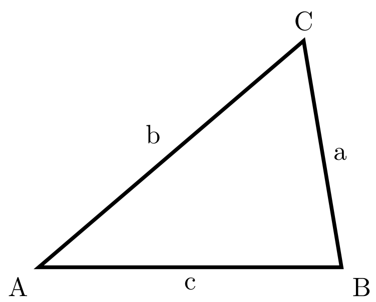
We've transformed the expression to a form where n appears only once, making the analysis much easier.

It's now clear that this expression will only be independent of n if either $d = 0$, which eliminates the nd term in the denominator, or if $d = 4$, which makes the numerator of the fraction zero, eliminating the whole fraction. As in method 1, since the question states the common difference is not zero, we conclude that it is 4.

It's also interesting to note that when $d = 4$ the above expression clearly collapses to 3, which matches the result from the check above. Questions for further investigation: What does it collapse to when $d = 0$? Does that result make sense?

Question 5

(a) The symbols, a, b and c are not defined in the question, so we'll assume that they are being used in the traditional sense encountered for Sine Rule and Cosine Rule. That is, they denote the lengths of the sides opposite angles A, B and C respectively.



We are given that $\angle B = 2A$. Since the angles of a triangle sum to $180°$, we know $\angle C = 180° - 3A$

We need to use both Sine Rule and Cosine Rule. This gives a choice of the following equations.

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin 2A}{b} = \frac{\sin(180° - 3A)}{c} \\
a^2 &= b^2 + c^2 - 2bc\cos A \\
b^2 &= a^2 + c^2 - 2ac\cos 2A \\
c^2 &= a^2 + b^2 - 2ab\cos(180° - 3A)
\end{aligned}$$

How do we choose which ones to use?

We are asked to prove $b^2 = a^2 + ac$. This bears some resemblance to the 2nd Cosine Rule equation, so that's a good starting point. The result does not refer to any angles, while the Cosine Rule equation contains $\cos 2A$, so we hope to use Sine Rule to replace $\cos 2A$ by something that only refers to edge lengths.

Looking at the Sine Rule equations, we can simplfy $\sin(180° - 3A)$ to $\sin 3A$, but the expansion of $\sin 3A$ is ugly, so we'd prefer to avoid using that term if possible. Let's see what we can do with the two simpler terms.

$$\begin{aligned}\frac{\sin A}{a} &= \frac{\sin 2A}{b}\\ &= \frac{2\sin A\cos A}{b}\end{aligned}$$

We know $\angle A > 0$, since every angle in a triangle is positive, and $\angle A + \angle B = 3A < 180°$, because the three angles of the triangle sum to $180°$. Hence $0 < A < 60°$. Hence $\sin A \neq 0$, which allows us to simplify the previous equation to

$$\begin{aligned}\frac{1}{a} &= \frac{2\cos A}{b}\\ \cos A &= \frac{b}{2a}\end{aligned}$$

It looks useful to convert the $\cos 2A$ term in the Cosine Rule to an expression involving $\cos A$, which can be replaced by the above expression.

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$$

Hence the 2nd form of Cosine Rule gives

$$\begin{aligned}b^2 &= a^2 + c^2 - 2ac\cos 2A\\ &= a^2 + c^2 - 2ac(2\cos^2 A - 1)\\ &= a^2 + c^2 - 2ac(2\frac{b^2}{4a^2} - 1)\\ &= a^2 + c^2 - \frac{b^2 c}{a} + 2ac\end{aligned}$$

This has a lot of the terms we want in the result, plus some extra terms that I can't see what to do with. Let's take the terms we want in the result to the left hand side and see what is left over.

$$b^2 - a^2 - ac = c^2 - \frac{b^2 c}{a} + ac$$

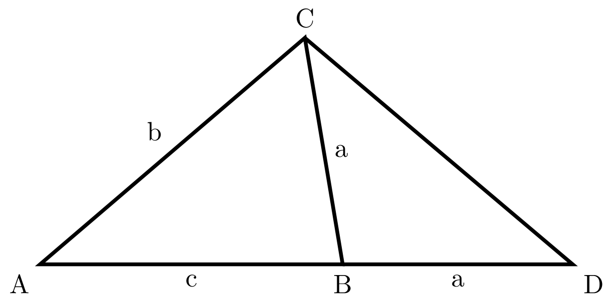
That has made things clearer. The right hand side is a multiple of the left hand side.

$$\begin{aligned}b^2 - a^2 - ac &= -\frac{c}{a}(b^2 - a^2 - ac)\\ (b^2 - a^2 - ac)(1 + \frac{c}{a}) &= 0\end{aligned}$$

Since a and c are positive $(1 + \frac{c}{a}) \neq 0$ so we can conclude

$$\begin{aligned} b^2 - a^2 - ac &= 0 \\ b^2 &= a^2 + ac \\ &= a(a+c) \end{aligned}$$

(b) Construction: Extend AB to D such that $BD = BC = a$



This next part will involve more than one triangle, so for clarity we will introduce a new symbol. Let $\angle A = \theta$.

In $\triangle ABC$:

$\angle A = \theta$

$\angle B = 2\angle A = 2\theta$

$\angle C = 180° - 3\theta$, because the angle sum of a triangle is 180°.

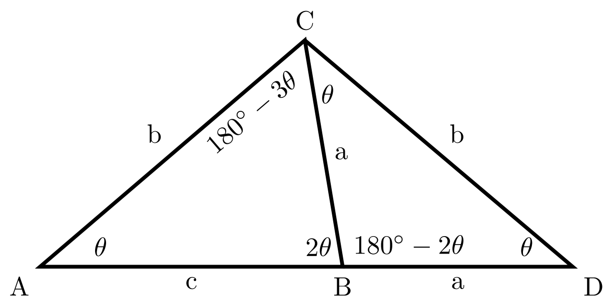
Since $\angle ABC + \angle CBD = 180°$ we have $\angle CBD = 180° - 2\theta$.

In $\triangle BCD, BC = BD$, so this triangle is either isosceles and so $\angle BCD = \angle BDC$, or it is equilateral, which also makes those two angles equal because all 3 angles would be equal. Since the three angles of the triangle sum to 180°, we conclude $\angle BCD = \angle BDC = \theta$.

(While not necessary for a valid proof, we can rule out the equilateral triangle case. $\triangle BCD$ can only be equilateral if $\theta = 60°$. But this would make $\angle ACB = 0$, so ABC would no longer be a triangle.)

$\angle CAD = \angle CDA$, so $\triangle ACD$ is either isosceles or equilateral. Either way, $CD = AC = b$. (Again, while not necessary for a valid proof, we can rule out the equilateral case, because $\theta \neq 60°$)

Here is an updated diagram showing the angles.



The next step is to locate the similar triangles. Two triangles are similar if their corresponding angles are equal. The similar triangles are likely to be the isosceles triangles, but if that isn't clear, just list the angles for all three available triangles to make it clearer.

In $\triangle ABC$:	$\angle A = \theta$	$\angle B = 2\theta$	$\angle C = 180° - 3\theta$
In $\triangle CDB$:	$\angle C = \theta$	$\angle D = \theta$	$\angle B = 180° - 2\theta$
In $\triangle ADC$:	$\angle A = \theta$	$\angle D = \theta$	$\angle C = 180° - 2\theta$

Hence the similar triangles are $\triangle CDB$ and $\triangle ADC$.

In similar triangles the ratio of the lengths of corresponding sides is constant. Therefore

$$\begin{aligned}\frac{CD}{BC} &= \frac{AD}{AC}\\ \frac{b}{a} &= \frac{a+c}{b}\\ b^2 &= a(a+c)\end{aligned}$$